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Global existence results for Oldroyd-B fluids in exterior domains

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ABSTRACT

In this paper we consider the set of equations describing Oldroyd-B fluids in exterior domains. It is shown that these equations admit a unique, global solution defined in a certain function space provided the initial data and the coupling constant are small enough.

1. Introduction and main result

Incompressible fluids are being described by the set of equations

$$\begin{cases} \varrho\big(u_t + (u\cdot\nabla)u\big) = \operatorname{div}\sigma + f, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

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where u denotes the velocity of the fluid, σ its stress tensor, ϱ its density and f an outer force. The stress tensor σ may be decomposed as $\sigma = -pId + \tau$, where p denotes the pressure of the fluid and τ the tangential part of the stress tensor.

In case of Newtonian fluids one has

$$\tau = 2\eta D(u),$$

where η denotes the viscosity of the fluid and $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ the deformation tensor.

For many type of fluids it is impossible to describe the tangential part $\tau(t)$ of the stress tensor $\sigma(t)$ at time t knowing only $D(u(t))$. One has hence to take into account in addition the history of $D(u)$. In this case, one says that the fluid has a "memory" and is of viscoelastic type. The Oldroyd model, see [17], is one of the classical models of viscoelastic fluids and here τ is being described by the differential equation

$$\tau_t + (u \cdot \nabla)\tau + b\tau + F(\tau, \nabla\tau) = 0,$$

where $b > 0$ and F is a quadratic form in $(\tau, \nabla u)$.

In this paper, we do not consider the general Oldroyd model with eight constants but the simpler case of so-called Oldroyd-B fluids. Here τ is being determined by the equation

$$\tau + \lambda_1 \frac{D_a\tau}{Dt} = 2\eta\left[D(u) + \lambda_2 \frac{D_a D(u)}{Dt}\right], \tag{1.2}$$

where $\frac{D_a}{Dt}$ denotes the "objective derivative" given by

$$\frac{D_a\tau}{Dt} = \tau_t + (v \cdot \nabla)\tau + g_a(\tau, \nabla u)$$

and g_a is given by

$$g_a(\tau, \nabla u) = \tau W(u) - W(u)\tau - a\big[D(u)\tau + \tau D(u)\big]$$

for some $a \in [-1, 1]$. Here $W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$ denotes the vorticity tensor; the parameters $\lambda_1, \lambda_2 \geqslant 0$ denote the relaxation and retardation time, respectively, and satisfy $\lambda_2 \leqslant \lambda_1$. Fluids of this type have viscous as well as elastic properties. Note that the case $\lambda_2 = \lambda_1 = 0$ corresponds to purely viscous fluids (being described by the Navier–Stokes equation), whereas the case $\lambda_1 > \lambda_2 = 0$ describes a purely elastic fluid.

Setting $\tau = \tau_N + \tau_E$ with

$$\tau_N = 2\eta \frac{\lambda_2}{\lambda_1} D(u),$$

it follows that τ_E satisfies

$$\tau_E + \lambda_1 \frac{D_a\tau_E}{Dt} = 2\eta\left(1 - \frac{\lambda_2}{\lambda_1}\right)D(u).$$

Setting with some abuse of notation $\tau = \tau_E$, the above set of Eqs. (1.1) and (1.2) may be rewritten in the form

$$\begin{cases} \varrho\big(u_t + (u \cdot \nabla)u\big) - \eta(1-\alpha)\Delta u + \nabla p = \operatorname{div}\tau + f, \\ \operatorname{div} u = 0, \\ \tau + \lambda_1\big(\tau_t + (u \cdot \nabla)\tau + g_a(\tau, \nabla u)\big) = 2\eta\alpha D(u), \end{cases} \tag{1.3}$$

where $\alpha = 1 - \lambda_2/\lambda_1$.

Assuming finally that $f = 0$, that the equations for the fluid are defined in a domain $\Omega \subset \mathbb{R}^n$, that the fluid is subject to Dirichlet boundary conditions and introducing dimensionless variables, we obtain

$$\begin{cases} \mathrm{Re}\big(u_t + (u \cdot \nabla)u\big) - (1-\alpha)\Delta u + \nabla p = \operatorname{div}\tau & \text{in } \Omega \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0,\infty), \\ \mathrm{We}\big(\tau' + (u \cdot \nabla)\tau\big) + \tau = 2\alpha D(u) - \mathrm{We} g_a(\tau, \nabla u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(0,\cdot) = u_0 & \text{in } \Omega, \\ \tau(0,\cdot) = \tau_0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where Re and We denote the Reynolds and Weissenberg number, respectively, of the fluid.

In the case where $\Omega \subset \mathbb{R}^n$ is a *bounded* domain with boundary of class C^3, for some $\mu > 0$, Guillopé and Saut [5] proved the existence of a unique, local, strong solution to Eq. (1.4) defined in appropriate Sobolev spaces $H^s(\Omega)$. Moreover, this solution exists on $[0,\infty)$ provided the data as well as the coupling between the two equations are sufficiently small; see [6]. Their result was extended to the L^p-setting by Fernández-Cara, Guillén and Ortega in [2]. More precisely, using maximal L^s-regularity for the Stokes operator A_r in $L^r_\sigma(\Omega)$ for $1 < s < \infty$ and $n < r < \infty$, they proved the existence of a unique, local, strong solution $u \in L^s(J; D(A_r))$, $\tau \in C(J : W^{1,r}(\Omega))$. Moreover, given $T > 0$, this solution can be continued until T, provided data and coupling are small enough in suitable norms.

Note that their approach relies essentially on the fact that the constant appearing in the maximal regularity estimate for the Stokes operator does *not* depend on T; see therefore the work of Giga and Sohr [4]. Hence, it seems that the approach described in [2] is thus restricted to the case of *bounded* domains.

It is the aim of this paper to extend the results of the above type to *exterior domains* within the Hilbert space setting. Note that a straightforward extension of the method described by Guillopé and Saut in [5] and [6] seems not be possible due to the lack of compactness for the mapping in question for unbounded domains. Using a local compactness method, Talhouk proved in [20] a *local* existence as well as a uniqueness result for such kind of fluids in a class of unbounded and uniform regular domains. In our main result we show that there exists a unique, *global* solution to Eq. (1.4) in *exterior* domains provided the data as well as the coupling are small enough.

Results on the stationary case are due to Renardy [18] who proved that if f is sufficiently small and $0 < \alpha \leqslant 1$, then there exists a unique small strong solution. For global existence results concerning shear flows of certain classes of viscoelastic fluids, we refer to [19].

Existence of global weak solutions has been proved by Lions and Masmoudi in [15] for $a = 0$ in the case of $\Omega = \mathbb{R}^n$. Chemin and Masmoudi [1] established existence and uniqueness results for local and global solutions in certain scaling invariant spaces of type $L^\infty_{\mathrm{loc}}([0,T); H^s(\mathbb{R}^n))$ for $s > n/2$. Also, necessary conditions for blow-up of the solution are given there. A new method for proving and improving the Chemin–Masmoudi criterion is presented by Lei, Masmoudi and Zhou in [10].

We also would like to point out that local and global well-posedness results in the situation of the whole space $\mathbb{R}^3$ or for *bounded* domains $\Omega \subset \mathbb{R}^3$ in the case where the Weissenberg number is considered to be *infinite*, were obtained by Lin, Liu and Zhang in [12], Lin, Liu and Zhou in [13] and Lin and Zhang in [14]. For further results concerning in particular the two-dimensional case via the incompressible limit approach and the strain-rotation decomposition, respectively, we refer to [11,8,9]. Note that our approach makes use of regularity properties of the stationary Stokes equation described in Proposition 2.3 below. For this reason we concentrate here on three or higher space dimensions.

To the best of our knowledge, the following theorem is the first result concerning Oldroyd-B fluids in *exterior* domains.

In order to formulate our main result, we introduce the well-known Stokes operator A defined in $L^2_\sigma(\Omega)$, Ω being an exterior domain with boundary of class C^3, as $Au := P\Delta u$ for all $u \in D(A) := H^2(\Omega) \cap H^1_0(\Omega) \cap L^2_\sigma(\Omega)$ as well as the space $V := H^1_0(\Omega) \cap L^2_\sigma(\Omega)$.

Then our main result reads as follows.

Theorem 1.1. *Let $J := [0, \infty)$, $n \geqslant 3$ and let $\Omega \subset \mathbb{R}^n$ be an exterior domain with boundary $\partial\Omega$ of class C^3. Then there exists $\varepsilon_0 > 0$ such that if*

$$\|u_0\|_{D(A)} + \|\tau_0\|_{H^2} \leqslant \varepsilon_0 \quad \text{and} \quad \alpha \leqslant \varepsilon_0,$$

then there exists a unique, global strong solution to Eq. (1.4) *for all $t \in J$ satisfying*

$$\begin{aligned}
&u \in L^\infty\big(J; D(A)\big) \quad \text{such that} \quad \nabla u \in L^2\big(J; H^2(\Omega)\big) \quad \text{and} \quad u' \in L^\infty\big(J; L^2_\sigma(\Omega)\big) \cap L^2(J; V),\\
&\nabla p \in L^\infty\big(J; H^1(\Omega)\big) \cap L^2\big(J; H^1(\Omega)\big),\\
&\tau \in L^\infty\big(J; H^2(\Omega)\big) \cap L^2\big(J; H^2(\Omega)\big) \quad \text{such that} \quad \tau' \in L^\infty\big(J; L^2(\Omega)\big) \cap L^2\big(J; L^2(\Omega)\big).
\end{aligned}$$

The following corollary follows from interpolation.

Corollary 1.2. *Under the assumptions of the above theorem, the solutions u and τ to* (1.4) *satisfy*

$$\begin{aligned}
&u \in C_b\big(J; D(A)\big) \quad \text{such that} \quad u' \in C_b\big(J; L^2_\sigma(\Omega)\big) \quad \text{and}\\
&\tau \in C_b\big(J; H^2(\Omega)\big) \quad \text{such that} \quad \tau' \in C_b\big(J; H^1(\Omega)\big).
\end{aligned}$$

2. Estimates for the decoupled linear equations

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary $\partial\Omega$ of class C^3 and as above let $J := [0, \infty)$. We start this section by introducing the space $L^2_\sigma(\Omega)$. To this end, set

$$\begin{aligned}
G_2(\Omega) &:= \big\{u \in L^2(\Omega)\colon\ u = \nabla\pi \text{ for some } \pi \in H^1_{\mathrm{loc}}(\Omega)\big\},\\
L^2_\sigma(\Omega) &:= \overline{\big\{u \in C^\infty_c(\Omega)\colon\ \operatorname{div} u = 0 \text{ in } \Omega\big\}}^{\|\cdot\|}.
\end{aligned}$$

Then $L^2(\Omega)$ can be decomposed into

$$L^2(\Omega) = L^2_\sigma(\Omega) \oplus G_2(\Omega),$$

and there exists a unique projection $P : L^2(\Omega) \to L^2_\sigma(\Omega)$ having $G_2(\Omega)$ as its null space. P is called the Helmholtz projection. It can be shown that

$$L^2_\sigma(\Omega) = \big\{u \in L^2(\Omega)\colon\ \operatorname{div} u = 0,\ \nu \cdot u_{|\partial\Omega} = 0\big\}, \tag{2.1}$$

where ν denotes the exterior normal to $\partial\Omega$. For a proof of this fact, we refer to Theorem 1.6 of [16]. We will use the following Green's formula for functions defined on an exterior domain Ω with smooth boundary $\partial\Omega$. Set $Y_2(\Omega) = \{u \in L^2(\Omega)^n\colon\ \operatorname{div} u \in L_2(\Omega)\}$. Then there exists a bounded operator $T_\nu : Y_2(\Omega) \to H^{-1/2}(\partial\Omega)$ such that $T_\nu u = u \cdot \nu$ on $\partial\Omega$ and

$$(T_\nu u, v_{|\partial\Omega})_{\partial\Omega} = (\operatorname{div} u, v)_\Omega + (u, \nabla v)_\Omega, \quad v \in H^1(\Omega). \tag{2.2}$$

For a proof of this fact we refer to Proposition 1.2 of [16]. The Stokes operator A in $L^2_\sigma(\Omega)$ is defined as

$$Au := P\Delta u \quad \text{for all } u \in D(A) := H^2(\Omega) \cap H^1_0(\Omega) \cap L^2_\sigma(\Omega).$$

We set further $V := H^1_0(\Omega) \cap L^2_\sigma(\Omega)$.

In the sequel, we will make use the following result on the linear Stokes equation. More precisely, consider the equation

$$\begin{cases} \mathrm{Re} u_t - (1-\alpha)\Delta u + \nabla p = F & \text{in } \Omega \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{2.3}$$

Adapting the arguments given in Chapter III.1 of [21] to the situation of exterior domains and smooth data, one obtains the following result.

Proposition 2.1. *Let $\partial\Omega \in C^3$, $u_0 \in D(A)$, $F \in L^2(J; H^1(\Omega))$ and $F' \in L^2(J; H^{-1}(\Omega))$. Then there exists a unique solution $(u, \nabla p)$ of Eq. (2.3) satisfying*

$$\begin{aligned} & u \in L^\infty\big(J; D(A)\big) \quad \textit{such that} \quad \nabla u \in L^2\big(J; H^2(\Omega)\big), \\ & u' \in L^2(J; V) \cap L^\infty\big(J; L^2_\sigma(\Omega)\big), \\ & \nabla p \in L^2\big(J; H^1(\Omega)\big) \cap L^\infty\big(J; H^1(\Omega)\big). \end{aligned}$$

This higher order regularity result for the Stokes equation motivates the introduction of the following Banach spaces

$$\begin{aligned} X_1 &:= \big\{u \in L^\infty\big(J; D(A)\big) \text{ such that } \nabla u \in L^2\big(J; H^2(\Omega)\big) \text{ and } u' \in L^\infty\big(J; L^2_\sigma(\Omega)\big) \cap L^2(J; V)\big\}, \\ X_2 &:= \big\{\nabla p \in L^\infty\big(J; H^1(\Omega)\big) \cap L^2\big(J; H^1(\Omega)\big)\big\}, \\ X_3 &:= \big\{\tau \in L^\infty\big(J; H^2(\Omega)\big) \cap L^2\big(J; H^2(\Omega)\big) \text{ such that } \tau' \in L^\infty\big(J; L^2(\Omega)\big) \cap L^2\big(J; L^2(\Omega)\big)\big\}, \end{aligned}$$

equipped with their natural norms

$$\begin{aligned} \|u\|_{X_1} &:= \|u\|_{L^\infty(J;D(A))} + \big\|u'\big\|_{L^\infty(J;L^2)} + \|\nabla u\|_{L^2(J;H^2)} + \big\|u'\big\|_{L^2(J;V)}, \\ \|\nabla p\|_{X_2} &:= \|\nabla p\|_{L^\infty(J;H^1)} + \|\nabla p\|_{L^2(J;H^1)}, \\ \|\tau\|_{X_3} &:= \|\tau\|_{L^\infty(J;H^2)} + \big\|\tau'\big\|_{L^\infty(J;L^2)} + \|\tau\|_{L^2(J;H^2)} + \big\|\tau'\big\|_{L^2(J;L^2)}. \end{aligned}$$

Furthermore, for $v \in X_1$ and $\theta \in X_3$ consider the transport equation

$$\begin{cases} \mathrm{We}\big(\tau_t + (v\cdot\nabla)\tau\big) + \tau = 2\alpha D(v) - \mathrm{We}\, g_a(\theta, \nabla v) & \text{in } \Omega \times (0,\infty), \\ \tau(0) = \tau_0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

By the method of characteristics, we have the following existence and uniqueness result for the transport equation.

Proposition 2.2. *Let $\tau_0 \in H^2(\Omega)$, $v \in X_1$ and $\theta \in X_3$. Then there exists a unique, strong solution $\tau \in X_3$ to* (2.4)*.*

We will also make use of the following higher order elliptic regularity result for the stationary Stokes system. For a proof, we refer to Theorem 4.1 of [3] or Lemma 4.3 of [7]. Observe that $\widehat{H}^j(\Omega)$ denotes the homogeneous Sobolev space of order j.

Proposition 2.3. *Let $m \in \{0, 1\}$ and $n \geqslant 3$. Let $\Omega \subset \mathbb{R}^n$ be an exterior domain with boundary of class C^{m+2} and $g \in H^m(\Omega)$. Then the equation*

$$\begin{cases} -\Delta u + \nabla p = g & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a solution $(u, p) \in \widehat{H}^{m+2}(\Omega) \times \widehat{H}^{m+1}(\Omega)$ which is unique provided $\nabla u \in L^2(\Omega)$. In this case, there exists a constant $C > 0$ such that

$$\left\|\nabla^2 u\right\|_{H^m} + \|\nabla p\|_{H^m} \leqslant C\big(\|g\|_{H^m} + \|\nabla u\|\big).$$

3. Proof of the main theorem

We start this section by defining the space X as

$$X := X_1 \times X_2 \times X_3,$$

equipped with its natural norm

$$\left\|(u, \nabla p, \tau)\right\|_X := \|u\|_{X_1} + \|\nabla p\|_{X_2} + \|\tau\|_{X_3}.$$

Furthermore, for $M > 0$, $u_0 \in D(A)$ and $\tau_0 \in H^2(\Omega)$ we set

$$X_M := \left\{(u, \nabla p, \tau) \in X_1 \times X_2 \times X_3\colon u(0) = u_0,\ \tau(0) = \tau_0,\ \left\|(u, \nabla p, \tau)\right\|_X \leqslant M\right\}.$$

Now, given $(v, \nabla q, \theta) \in X$, we define the mapping

$$\Phi(v, \nabla q, \theta) := (u, \nabla p, \tau),$$

where τ is defined to be the unique solution of Eq. (2.4) and $(u, \nabla p)$ is defined to be the unique solution of (2.3) with right-hand side $F = \operatorname{div}\tau - \mathrm{Re}(v \cdot \nabla)v$.

In the following, we show that Φ is a mapping from X to X. To this end, consider the set of equations

$$\begin{cases} \mathrm{Re} u_t - (1-\alpha)\Delta u + \nabla p = \nabla \cdot \tau - \mathrm{Re}(v \cdot \nabla)v & \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, \infty), \\ \mathrm{We}\big(\tau_t + (v \cdot \nabla)\tau\big) + \tau = 2\alpha D(v) - \mathrm{We} g_a(\theta, \nabla v) & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u_0 & \text{in } \Omega, \\ \tau(0) = \tau_0 & \text{in } \Omega. \end{cases} \tag{3.1}$$

Then the following holds.

Proposition 3.1. *Let $(v, \theta) \in X_1 \times X_3$. Then Eq.* (3.1) *admits a unique solution $(u, \nabla p, \tau) \in X$ and there exists a constant $C > 0$ such that*

$$\left\|(u, \nabla p, \tau)\right\|_X \leqslant C\big(\|\tau_0\|_{H^2} + \|u_0\|_{D(A)} + \alpha\left\|(v, \nabla q, \theta)\right\|_X + \left\|(v, \nabla q, \theta)\right\|_X^2\big).$$

Proof. We subdivide the proof into two steps concerning estimates for the transport and the fluid equation, respectively.

Step 1: *Estimates for the transport equation.*

Multiplying the third equation of (3.1) by τ yields

$$\begin{aligned}
&\frac{\mathrm{We}}{2}\frac{d}{dt}\|\tau(t)\|^2+\mathrm{We}\big(\big(v(t)\cdot\nabla\big)\tau(t),\tau(t)\big)+\|\tau(t)\|^2\\
&\quad\leqslant \|2\alpha D(v)(t)\|\,\|\tau(t)\|+4\mathrm{We}\|\theta(t)\nabla v(t)\|\,\|\tau(t)\|\\
&\quad\leqslant 2\alpha^2\|\nabla v(t)\|^2+4\mathrm{We}^2\|\theta(t)\nabla v(t)\|^2+\frac{1}{2}\|\tau(t)\|^2.
\end{aligned}$$

Moreover, by Hölder's inequality and by Sobolev embeddings, there exists a constant $C>0$ such that

$$\|\theta(t)\nabla v(t)\|\leqslant C\|\theta(t)\|_3\|\nabla v(t)\|_6\leqslant C\|\theta(t)\|_{H^1}\|\nabla v(t)\|_{H^1}.$$

Since $v(t)\in L^2_\sigma(\Omega)$ for all $t\in J$, it follows from (2.1) and (2.2) that $((v(t)\cdot\nabla)\tau(t),\tau(t))=0$ for all $t>0$. Integrating, we obtain

$$\begin{aligned}
\mathrm{We}\|\tau(t)\|^2+\int_0^t\|\tau(s)\|^2\,ds&\leqslant C\mathrm{We}^2\int_0^t\|\theta(s)\|_{H^1}^2\|\nabla v(s)\|_{H^1}^2\,ds+\mathrm{We}\|\tau_0\|^2\\
&\quad+C\alpha^2\int_0^t\|\nabla v(s)\|^2\,ds,\quad t>0.
\end{aligned}\tag{3.2}$$

Next, multiplying the time derivative of the third equation with τ_t yields

$$\begin{aligned}
&\frac{\mathrm{We}}{2}\frac{d}{dt}\|\tau_t(t)\|^2+\mathrm{We}\big(\big(v_t(t)\cdot\nabla\big)\tau(t),\tau_t(t)\big)+\mathrm{We}\big(\big(v(t)\cdot\nabla\big)\tau_t(t),\tau_t(t)\big)+\|\tau_t(t)\|^2\\
&\quad=\big(2\alpha D\big(v_t(t)\big),\tau_t(t)\big)-\mathrm{We}\big(g_a(\theta_t,\nabla v)(t),\tau_t(t)\big)-\mathrm{We}\big(g_a(\theta,\nabla v_t)(t),\tau_t(t)\big).
\end{aligned}$$

Thus, for $\varepsilon>0$ and $t>0$

$$\begin{aligned}
\frac{\mathrm{We}}{2}\frac{d}{dt}\|\tau_t(t)\|^2+\|\tau_t(t)\|^2&\leqslant\frac{\mathrm{We}^2}{4\varepsilon}\|v_t(t)\nabla\tau(t)\|^2+\varepsilon\|\tau_t(t)\|^2+\frac{(2\alpha)^2}{4\varepsilon}\|\nabla v_t(t)\|^2+\varepsilon\|\tau_t(t)\|^2\\
&\quad+\frac{\mathrm{We}^2}{4\varepsilon}\|4\theta_t(t)\nabla v(t)\|^2+\varepsilon\|\tau_t(t)\|^2+\frac{\mathrm{We}^2}{4\varepsilon}\|4\theta(t)\nabla v_t(t)\|^2\\
&\quad+\varepsilon\|\tau_t(t)\|^2.
\end{aligned}$$

The following inequalities induced by Sobolev's embeddings

$$\begin{aligned}
\|v_t(t)\nabla\tau(t)\|&\leqslant\|v_t(t)\|_3\|\nabla\tau(t)\|_6\leqslant C\|v_t(t)\|_{H^1}\|\nabla\tau(t)\|_{H^1},\\
\|\theta_t(t)\nabla v(t)\|&\leqslant\|\theta_t(t)\|\,\|\nabla v(t)\|_\infty\leqslant C\|\theta_t(t)\|\,\|\nabla v(t)\|_{H^2},\\
\|\theta(t)\nabla v_t(t)\|&\leqslant\|\theta(t)\|_\infty\|\nabla v_t(t)\|\leqslant C\|\theta(t)\|_{H^2}\|\nabla v_t(t)\|,\quad t>0,
\end{aligned}\tag{3.3}$$

imply

$$\mathrm{We}\big\|\tau_t(t)\big\|^2+\int_0^t\big\|\tau_s(s)\big\|^2\,ds\leqslant C\Bigg[\mathrm{We}^2\int_0^t\big\|v_s(s)\big\|_{H^1}^2\big\|\nabla\tau(s)\big\|_{H^1}^2\,ds+\alpha^2\big\|\nabla v_s(s)\big\|^2$$

$$+\mathrm{We}^2\big\|\theta_s(s)\big\|^2\big\|\nabla v(s)\big\|_{H^2}^2+\mathrm{We}^2\big\|\theta(t)\big\|_{H^2}^2\big\|\nabla v_s(s)\big\|^2\,ds\Bigg]$$

$$+\mathrm{We}\big\|\tau_t(0)\big\|^2. \tag{3.4}$$

Next, multiplying the space derivative of the third equation by $\nabla\tau$, applying (3.3) and integrating with respect to t, we obtain

$$\mathrm{We}\big\|\nabla\tau(t)\big\|^2+\int_0^t\big\|\nabla\tau(s)\big\|^2\,ds\leqslant C\int_0^t\big[\mathrm{We}^2\big\|\nabla v(s)\big\|_{H^1}^2\big\|\nabla\tau(s)\big\|_{H^1}^2\,ds+\alpha^2\big\|\nabla v(s)\big\|_{H^1}^2$$

$$+\mathrm{We}^2\big\|\nabla\theta(s)\big\|_{H^1}^2\big\|\nabla^2 v(s)\big\|^2+\mathrm{We}^2\big\|\theta(t)\big\|_{H^2}^2\big\|\nabla^2 v(s)\big\|^2\big]\,ds$$

$$+\mathrm{We}\|\nabla\tau_0\|^2. \tag{3.5}$$

Finally, we multiply the second space derivatives of the third equation by $\nabla^2\tau$ and obtain

$$\mathrm{We}\big\|\nabla^2\tau(t)\big\|^2+\int_0^t\big\|\nabla^2\tau(s)\big\|^2\,ds\leqslant C\int_0^t\big[\mathrm{We}^2\big\|\nabla v(s)\big\|_{H^2}^2\big\|\nabla\tau(s)\big\|_{H^1}^2+\alpha^2\big\|\nabla^3 v(s)\big\|^2$$

$$+\mathrm{We}^2\big\|\theta(s)\big\|_{H^2}^2\big\|\nabla v(s)\big\|_{H^2}^2\big]\,ds+\mathrm{We}\big\|\nabla^2\tau_0\big\|^2,\quad t>0. \tag{3.6}$$

Summing up the estimates (3.2), (3.4), (3.5) and (3.6), we notice that there exists a constant $C>0$ such that

$$\mathrm{We}\big\|\tau(t)\big\|_{H^2}^2+\mathrm{We}\big\|\tau_t(t)\big\|^2+\int_0^t\big(\big\|\tau(s)\big\|_{H^2}^2+\big\|\tau_s(s)\big\|^2\big)\,ds$$

$$\leqslant C\mathrm{We}^2\Bigg[\int_0^t\frac{\alpha^2}{\mathrm{We}^2}\big\|\nabla v(s)\big\|_{H^2}^2+\frac{\alpha^2}{\mathrm{We}^2}\big\|\nabla v_s(s)\big\|^2+\big\|\theta(s)\big\|_{H^2}^2\big\|\nabla v(s)\big\|_{H^2}^2+\big\|\theta_s(s)\big\|^2\big\|\nabla v(s)\big\|_{H^2}^2$$

$$+\big\|\theta(s)\big\|_{H^2}^2\big\|\nabla v_s(s)\big\|^2+\big\|\nabla^2\tau(s)\big\|^2\big\|v_s(s)\big\|_{H^1}^2+\big\|\nabla\tau(s)\big\|_{H^1}^2\big\|\nabla v(s)\big\|_{H^2}^2\,ds\Bigg]$$

$$+\mathrm{We}\big(\|\tau_0\|_{H^2}^2+\big\|\tau_t(0)\big\|^2\big),\quad t>0. \tag{3.7}$$

Hence,

$$\begin{aligned}
&\mathrm{We}\|\tau\|_{L^\infty(J;H^2)}^2+\big\|\tau'\big\|_{L^\infty(J;L^2)}^2+\mathrm{We}^2\|\tau\|_{L^2(J;H^2)}^2+\big\|\tau'\big\|_{L^2(J;L^2)}^2\\
&\quad\leqslant C\alpha^2\big(\|\nabla v\|_{L^2(J;H^2)}^2+\big\|v'\big\|_{L^2(J;V)}^2\big)+C\mathrm{We}^2\big(\|\theta\|_{L^\infty(J;H^2)}^2+\big\|\theta'\big\|_{L^\infty(J;L^2)}^2\big)\|\nabla v\|_{L^2(J;H^2)}^2\\
&\qquad+\|\theta\|_{L^\infty(J;H^2)}^2\big\|v'\big\|_{L^2(J;V)}^2+C\mathrm{We}^2\big(\|\tau\|_{L^\infty(J;H^2)}^2\big[\big\|v'\big\|_{L^2(J;V)}^2+\|\nabla v\|_{L^2(J;H^2)}^2\big]\big)\\
&\qquad+\mathrm{We}\big(\|\tau_0\|_{H^2}^2+\alpha^2\|u_0\|_V^2\big),
\end{aligned}$$

and thus there exists $C > 0$ such that

$$\|\tau\|_{X_3} \leqslant C\big[\alpha\big\|(v, \nabla q, \theta)\big\|_X + \big\|(v, \nabla q, \theta)\big\|_X^2 + \mathrm{We}\big(\|\tau_0\|_{H^2} + \alpha\|u_0\|_{H_1}\big)\big]. \tag{3.8}$$

Step 2: *Estimates for the fluid equations.*

Next, we consider estimates for the fluid equations in (3.1). To this end, we multiply the first line of (3.1) with u and obtain

$$\frac{\mathrm{Re}}{2}\frac{d}{dt}\big\|u(t)\big\|^2 + (1-\alpha)\big\|\nabla u(t)\big\|^2 = -\big(\tau(t), \nabla u(t)\big) - \mathrm{Re}\big(\big(v(t)\cdot\nabla\big)v(t), u(t)\big),$$

where we used the fact that $u(t) \in L^2_\sigma(\Omega)$ for all $t \in J$. By Sobolev's embedding theorems and Hölder's inequality, we obtain for $t > 0$

$$\begin{aligned}\big|\big(\big(v(t)\cdot\nabla\big)v(t), u(t)\big)\big| &\leqslant \big\|v(t)\big\|_3\big\|\nabla v(t)\big\|\big\|u(t)\big\|_6 \leqslant C\big\|v(t)\big\|_{H^1}\big\|\nabla v(t)\big\|\big\|\nabla u(t)\big\| \\ &\leqslant \frac{C^2}{1-\alpha}\big\|v(t)\big\|_{H^1}^2\big\|\nabla v(t)\big\|^2 + \frac{1-\alpha}{4}\big\|\nabla u(t)\big\|^2, \\ \big|\big(\tau(t), \nabla u(t)\big)\big| &\leqslant \frac{1}{1-\alpha}\big\|\tau(t)\big\|^2 + \frac{1-\alpha}{4}\big\|\nabla u(t)\big\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathrm{Re}\big\|u(t)\big\|^2 + (1-\alpha)\int_0^t \big\|\nabla u(s)\big\|^2\,ds \\ &\quad\leqslant 2\mathrm{Re}\|u_0\|^2 + C\int_0^t \big[\mathrm{Re}^2\big\|v(s)\big\|_{H^1}^2\big\|\nabla v(s)\big\|^2 + \big\|\tau(s)\big\|^2\big]\,ds, \quad t > 0.\end{aligned} \tag{3.9}$$

Next, consider the time derivative of the first equation of (3.1). Multiplying with u' and noting that $u' \in L^2(J; V)$, we have

$$\begin{aligned}\frac{\mathrm{Re}}{2}\frac{d}{dt}\big\|u_t(t)\big\|^2 + (1-\alpha)\big\|\nabla u_t(t)\big\|^2 &= -\big(\tau_t(t), \nabla u_t(t)\big) - \mathrm{Re}\big(\big\{\big(v(t)\cdot\nabla\big)v(t)\big\}_t, u_t(t)\big) \\ &\leqslant \frac{1}{1-\alpha}\big\|\tau_t(t)\big\|^2 + \frac{1-\alpha}{4}\big\|\nabla u_t(t)\big\|^2 \\ &\quad + \mathrm{Re}\bigg[\frac{C}{1-\alpha}\big\|\big\{(v\cdot\nabla)v\big\}_t(t)\big\|_{\frac{6}{5}}^2 + \frac{1-\alpha}{4}\big\|\nabla u_t(t)\big\|^2\bigg].\end{aligned}$$

Employing the inequality

$$\begin{aligned}\big\|\big\{(v\cdot\nabla)v\big\}_t(t)\big\|_{\frac{6}{5}} &\leqslant \big\|v_t(t)\nabla v(t)\big\|_{\frac{6}{5}} + \big\|v(t)\nabla v_t(t)\big\|_{\frac{6}{5}} \leqslant \big\|v_t(t)\big\|_3\big\|\nabla v(t)\big\| + \big\|v(t)\big\|_3\big\|\nabla v_t(t)\big\| \\ &\leqslant C\big\|v_t(t)\big\|_{H^1}\big\|\nabla v(t)\big\| + \big\|v(t)\big\|_{H^1}\big\|\nabla v_t(t)\big\|,\end{aligned}$$

yields

$$\mathrm{Re}\left\|u_t(t)\right\|^2+(1-\alpha)\int_0^t\left\|\nabla u_s(s)\right\|^2ds\leqslant 2\mathrm{Re}\left\|u_t(0)\right\|^2+C\int_0^t\Big[\mathrm{Re}^2\left\|v_s(s)\right\|_{H^1}^2\left\|\nabla v(s)\right\|^2$$
$$+\mathrm{Re}^2\left\|v(s)\right\|_{H^1}^2\left\|\nabla v_s(s)\right\|^2+\left\|\tau_s(s)\right\|^2\Big]ds. \qquad (3.10)$$

Multiplying the first equation of (3.1) by u_t and noting that $u_t\in L^2(J;V)$ yields

$$\mathrm{Re}\left\|u_t(t)\right\|^2+(1-\alpha)\frac{d}{dt}\left\|\nabla u(t)\right\|^2\leqslant\frac{1}{\mathrm{Re}}\left\|\nabla\tau(t)\right\|^2+\frac{\mathrm{Re}}{4}\left\|u_t(t)\right\|^2+\mathrm{Re}\|v\cdot\nabla v\|^2+\frac{\mathrm{Re}}{4}\left\|u_t(t)\right\|^2.$$

Integrating, we obtain

$$(1-\alpha)\left\|\nabla u(t)\right\|^2+\frac{\mathrm{Re}}{2}\int_0^t\left\|u_s(s)\right\|^2ds$$
$$\leqslant(1-\alpha)\|\nabla u_0\|^2+C\int_0^t\left(\left\|\nabla\tau(s)\right\|^2+\left\|v(s)\right\|_{H^1}^2\left\|\nabla v(s)\right\|^2\right)ds,\quad t>0. \qquad (3.11)$$

In order to estimates higher order derivatives of u and p we employ Proposition 2.3 and obtain

$$(1-\alpha)\left\|\nabla^2u(t)\right\|^2+\left\|\nabla p(t)\right\|^2+(1-\alpha)\int_0^t\left\|\nabla^2u(s)\right\|^2ds+\int_0^t\left\|\nabla p(s)\right\|^2ds$$
$$\leqslant\frac{C}{1-\alpha}\Bigg[\left\|\nabla\cdot\tau(t)\right\|^2+\mathrm{Re}^2\left\|v(t)\right\|_{H^1}^2\left\|\nabla^2v(t)\right\|^2+\mathrm{Re}^2\left\|u_t(t)\right\|^2+\left\|\nabla u(t)\right\|^2$$
$$+\int_0^t\left(\left\|\nabla\cdot\tau(s)\right\|^2+\mathrm{Re}^2\left\|v(s)\right\|_{H^1}^2\left\|\nabla^2v(s)\right\|^2+\mathrm{Re}^2\left\|u_s(s)\right\|^2+\left\|\nabla u(s)\right\|^2\right)ds\Bigg]. \qquad (3.12)$$

Next we apply Proposition 2.3 to the gradient of the fluid equation of (3.1) and obtain

$$(1-\alpha)\int_0^t\left\|\nabla^3u(s)\right\|^2ds+\int_0^t\left\|\nabla^2p(s)\right\|^2ds\leqslant\frac{C}{1-\alpha}\Bigg[\int_0^t\left(\left\|\nabla\cdot\tau(s)\right\|_{H^1}^2+\left\|\nabla v(s)\right\|_{H^1}^2\left\|\nabla^3v(s)\right\|^2\right.$$
$$\left.+\left\|u_s(s)\right\|_V^2+\left\|\nabla u(s)\right\|^2\right)ds\Bigg]. \qquad (3.13)$$

Taking into account the estimates (3.9) and (3.11), we see that

$$\|u\|_{L^\infty(J;H^1)}^2\leqslant C\big[\|u_0\|_{H^1}^2+\|\nabla v\|_{L^\infty(J;L^2)}^2\|v\|_{L^2(J;H^1)}^2+\|\tau\|_{L^2(J;H^1)}^2\big].$$

Moreover, by (3.12), (3.10) and (3.11)

$$
\begin{aligned}
\|\nabla^2 u\|^2_{L^\infty(J;L^2)} + \|\nabla p\|^2_{L^\infty(J;L^2)} \leqslant C\big[&\|\tau\|^2_{L^\infty(J;H^1)} + \|v\|^2_{L^\infty(J;H^1)}\|v\|^2_{L^\infty(J;H^2)} \\
&+ \|\nabla v\|^2_{L^\infty(J;L^2)}\|v'\|^2_{L^2(J;V)} + \|v\|^2_{L^\infty(J;H^1)}\|v'\|^2_{L^2(J;L^2)} \\
&+ \|\tau'\|^2_{L^2(J;L^2)} + \|\tau\|^2_{L^2(J;H^1)} + \|v\|^2_{L^\infty(J;H^1)}\|v\|^2_{L^2(J;H^1)} \\
&+ \|u_t(0)\|^2 + \|\nabla u_0\|^2\big].
\end{aligned} \tag{3.14}
$$

We next consider the term $\|\nabla u\|_{L^2(J;H^2)}$. Notice first that by (3.9)

$$
\|\nabla u\|^2_{L^2(J;L^2)} \leqslant C\big[\|u_0\|^2 + \|v\|^2_{L^\infty(J;H^1)}\|\nabla v\|^2_{L^2(J;L^2)} + \|\tau\|^2_{L^2(J;L^2)}\big].
$$

Further, by (3.12), (3.9) and (3.11)

$$
\begin{aligned}
\|\nabla^2 u\|^2_{L^2(J;L^2)} \leqslant C\big[&\|\tau\|^2_{L^2(J;H^1)} + \|v\|^2_{L^\infty(J;H^1)}\|\nabla^2 v\|^2_{L^2(J;L^2)} \\
&+ \|\nabla\tau\|^2_{L^2(J;L^2)} + \|v\|^2_{L^\infty(J;H^1)}\|\nabla v\|^2_{L^2(J;L^2)} + \|\tau'\|^2_{L^2(J;L^2)} \\
&+ \|v\|^2_{L^\infty(J;H^1)}\|v'\|^2_{L^2(J;V)} + \|\nabla u(0)\|^2 + \|u_0\|^2\big].
\end{aligned} \tag{3.15}
$$

Similarly,

$$
\begin{aligned}
\|\nabla^3 u\|^2_{L^2(J;L^2)} + \|\nabla^2 p\|^2_{L^2(J;L^2)} \leqslant C\big[&\|\tau\|^2_{L^2(J;H^2)} + \|\nabla v\|^2_{L^\infty(J;H^1)}\|\nabla v\|^2_{L^2(J;H^2)} \\
&+ \|u'\|^2_{L^2(J;V)} + \|\nabla^2 u\|^2_{L^2(J;L^2)} + \|\nabla u(0)\|^2 + \|u_0\|^2 \\
&+ \|u_0\|^2_{D(A)} + \|\tau_0\|^2_{H^1} + \|v\|^2_{L^\infty(J;L^2)}\|v\|^2_{L^\infty(J;H^1)}\big].
\end{aligned} \tag{3.16}
$$

Finally, we estimate the terms $\|u'\|_{L^2(J;V)}$ and $\|u'\|_{L^\infty(J;L^2)}$. It follows from (3.10) that

$$
\begin{aligned}
\|u'\|^2_{L^2(J;V)} \leqslant C\big[&\|u_0\|^2_{D(A)} + \|\tau_0\|^2_{H^1} + \|v\|^2_{L^\infty(J;L^2)}\|v\|^2_{L^\infty(J;H^1)} + \|v\|^2_{L^\infty(J;H^1)}\|v'\|^2_{L^2(J;V)} \\
&+ \|v\|^2_{L^\infty(J;H^1)}\|v'\|^2_{L^2(J;V)} + \|\tau'\|^2_{L^2(J;L^2)}\big],
\end{aligned} \tag{3.17}
$$

and that

$$
\begin{aligned}
\|u'\|^2_{L^\infty(J;L^2)} \leqslant C\big[&\|u_0\|^2_{D(A)} + \|\tau_0\|^2_{H^1} + \|v\|^2_{L^\infty(J;L^2)}\|v\|^2_{L^\infty(J;H^1)} + \|\tau'\|^2_{L^2(J;L^2)} \\
&+ \|\nabla v\|^2_{L^\infty(J;L^2)}\|v'\|^2_{L^2(J;V)} + \|v\|^2_{L^\infty(J;H^1)}\|v'\|^2_{L^2(J;V)}\big].
\end{aligned} \tag{3.18}
$$

Taking into account the estimates (3.9)–(3.18) as well as (3.8), we see that there exists a constant $C > 0$ such that

$$
\begin{aligned}
\|u\|_{X_1} + \|\nabla p\|_{X_2} \leqslant C\big[&\alpha\|(v, \nabla q, \theta)\|_X + \|(v, \nabla q, \theta)\|^2_X \\
&+ \mathrm{We}\big(\|\tau_0\|_{H^2} + \alpha\|u_0\|_{H_1} + \|u_0\|_{D(A)}\big)\big]. \qquad \square
\end{aligned} \tag{3.19}
$$

Remark 3.2. It follows from the above proposition that Φ maps X_M into X_M provided the norms of the initial data u_0 and τ_0, the coupling constant α and M are small enough. More precisely, there exists $\varepsilon_0 > 0$ such that if $M \leqslant \varepsilon_0$, $\|u_0\|_{D(A)} + \|\tau_0\|_{H^2} \leqslant \varepsilon_0$ and $\alpha \leqslant \varepsilon_0$, then Φ is a map from X_M into X_M.

We finally show that in addition $\Phi : X_M \to X_M$ is a contraction provided α and M are small enough.

Lemma 3.3. *There exists $\varepsilon_0 > 0$ such that if $M \leqslant \varepsilon_0$, $\|u_0\|_{D(A)} + \|\tau_0\|_{H^2} \leqslant \varepsilon_0$ and $\alpha \leqslant \varepsilon_0$, then $\Phi : X_M \to X_M$ is a contraction.*

Proof. Let $(v_1, \nabla q_1, \theta_1) \in X_M$, $(v_2, \nabla q_2, \theta_2) \in X_M$ and let $(u_1, \nabla p_1, \tau_1) = \Phi(v_1, \nabla q_1, \theta_1)$ and $(u_2, \nabla p_2, \tau_2) = \Phi(v_2, \nabla q_2, \theta_2)$. Moreover, set $u := u_1 - u_2$, $\nabla p := \nabla p_1 - \nabla p_2$, $\tau := \tau_1 - \tau_2$. Then $(u, \nabla p, \tau)$ is satisfying the following set of equation

$$\begin{cases} \mathrm{Re} u_t - (1-\alpha)\Delta u + \nabla p = \nabla \cdot \tau - \mathrm{Re}\big[(v_1 \cdot \nabla)v_1 - (v_2 \cdot \nabla)v_2\big] & \text{in } \Omega \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0,\infty), \\ \mathrm{We}\big(\tau_t + (v_1 \cdot \nabla)\tau_1 - (v_2 \cdot \nabla)\tau_2\big) + \tau & \\ \quad = 2\alpha D(v) - \mathrm{We}\big[g_a(\theta_1, \nabla v_1) - g_a(\theta_2, \nabla v_2)\big] & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(0) = 0 & \text{in } \Omega, \\ \tau(0) = 0 & \text{in } \Omega. \end{cases} \tag{3.20}$$

Using the identities

$$(v_1 \cdot \nabla)v_1 - (v_2 \cdot \nabla)v_2 = \frac{1}{2}\big[v_1 \nabla(v_1 - v_2) + v_2 \nabla(v_1 - v_2) + (v_1 - v_2)\nabla(v_1 + v_2)\big],$$
$$\theta_1 \nabla v_1 - \theta_2 \nabla v_2 = \frac{1}{2}\big[\theta_1 \nabla(v_1 - v_2) + \theta_2 \nabla(v_1 - v_2) + (\theta_1 - \theta_2)\nabla(v_1 + v_2)\big],$$

we see by the arguments given in the proof of Proposition 3.1 that

$$\begin{aligned} \big\|(u_1, \nabla p_1, \tau_1) - (u_2, \nabla p_2, \tau_2)\big\|_X \leqslant C\Big[&\big\|(v_1, \nabla q_1, \theta_1)\big\|_X \big\|\big(v_1 - v_2, \nabla(q_1 - q_2), \theta_1 - \theta_2\big)\big\|_X \\ &+ \big\|(v_2, \nabla q_2, \theta_2)\big\|_X \big\|\big(v_1 - v_2, \nabla(q_1 - q_2), \theta_1 - \theta_2\big)\big\|_X \\ &+ \alpha \big\|\big(v_1 - v_2, \nabla(q_1 - q_2), \theta_1 - \theta_2\big)\big\|_X\Big]. \end{aligned}$$

Finally, choosing M and α so small that $2CM + C\alpha \leqslant 1/2$, it follows that

$$\big\|(u_1, \nabla p_1, \tau_1) - (u_2, \nabla p_2, \tau_2)\big\|_{X_M} \leqslant \frac{1}{2}\big\|(v_1, \nabla q_1, \theta_1) - (v_2, \nabla q_2, \theta_2)\big\|_{X_M}. \qquad \square$$

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